SECOND DERIVATIVES INVOLVING PARAMETRIC EQUATIONS AND IMPLICIT DIFFERENTIATION

Example 1: Find $\frac{d^2y}{dx^2}$ for the curve with $x = t^2$ and $y = t + 4$.

The rule for finding the second derivative of parametric equations is:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

i.e. differentiate the first derivative $(\frac{dy}{dx})$ with respect to t, then multiply it with $\frac{dt}{dx}$.

- The first derivative is found using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 1 \times \frac{1}{2t} = \frac{1}{2t}$
- Differentiate $\frac{1}{2t}$ again with respect to t: $\frac{d}{dt}\left(\frac{1}{2}t^{-1}\right) = \frac{-1}{2}t^{-2} = \frac{-1}{2t^2}$
- So $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{-1}{2t^2} \times \frac{1}{2t} = \frac{-1}{4t^3}$

Example 2: Find $\frac{d^2y}{dx^2}$ of $4x^2 - 3y^2 = 9$.

- Find the first derivative using implicit differentiation: $8x - 6y\frac{dy}{dx} = 0$.
- Rearrange to make $\frac{dy}{dx}$ the subject: $\frac{dy}{dx} = \frac{8x}{6y} = \frac{4x}{3y}$
- Since the first derivative $\frac{dy}{dx}$ is a quotient, i.e. $\frac{f(x)}{g(x)}$ or in this case here $\frac{f(x)}{g(y)}$, we must use the quotient rule to differentiate, and also differentiate implicitly, to get the second derivative.
- Quotient Rule: $\frac{d^2y}{dx^2} = \frac{f'.g-g'.f}{g^2} = \frac{4.3y-3\frac{dy}{dx}.4x}{(3y)^2} = \frac{12y-12x\frac{dy}{dx}}{9y^2}$
- We need to substitute the term $\frac{dy}{dx}$ with $\frac{4x}{3y}$ and then simplify:

  $= \frac{12y-12x(\frac{4x}{3y})}{9y^2} = \frac{12y-\frac{16x^2}{y}}{9y^2} = \frac{12y^2-16x^2}{9y^2}$ (from multiplying top and bottom by y)
- Therefore $\frac{d^2y}{dx^2} = \frac{12y^2-16x^2}{9y^2}$.

Delta Ex 15.3 pg 152, Question 2